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Lattice sums of generalized hypergeometric functions

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Abstract. Representations and convergence criteria for one-dimensional (1D) and two-dimensional (2D) lattice sums of generalized hypergeometric functions ${}_pF_{p+1}$ are derived by using apparently new integral representations for the latter functions, and 1D and 2D forms of the Poisson summation formula. These lattice sum representations concisely unify and generalize various specializations given previously by several authors, and should prove useful in analysing finite-size effects in systems subjected to non-periodic boundary conditions.

1. Introduction

We consider the two-dimensional (2D) lattice sums of the generalized hypergeometric functions ${}_pF_{p+1}$ $(p \geqslant 0)$ defined for $x > 0$ by

$$W(\alpha, \beta; x) \equiv \sum_{m\in Z}\sum_{n\in Z}(-1)^{m\alpha+n\beta}\, {}_pF_{p+1}[(a_p); (b_{p+1}); -(m^2+n^2)x^2] \qquad (1.1a)$$

and

$$R(\alpha, \beta; x) \equiv \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{m\alpha+n\beta}\, {}_pF_{p+1}[(a_p); (b_{p+1}); -(m^2+n^2)x^2] \qquad (1.1b)$$

where $(\alpha, \beta) \in \{(1, 1), (0, 0), (1, 0), (0, 1)\}$, and Z is the set of all integers (positive, negative and zero). For conciseness in the following we define

$$\Delta \equiv \sum_{k=1}^{p+1} b_k - \sum_{k=1}^{p} a_k. \qquad (1.1c)$$

Several authors (see [1–3, 5, 6, 11]) have considered and found essentially elementary algebraic representations for the latter two lattice sums specialized by replacing the generalized hypergeometric function by either the Bessel function of the first kind (i.e. set $p = 0, b_1 = 1+\nu$) or by the trigonometric sine function (i.e. set $p = 0$, $b_1 = \frac{3}{2}$). Moreover, in [1, 7, 11] representations for these specializations of $W(\alpha, \beta; x)$ extended to higher dimensions were also deduced. All of the work just alluded to was motivated by problem 92-11 in SIAM Review which was proposed by Henkel and Weston in connection with finite-size scaling of the three-dimensional spherical model of ferromagnetism (see the editorial note in [2]). Aside from the interesting mathematical properties of the lattice sums being considered, Allen and Pathria [1] have noted that representations for such sums and their higher-dimensional generalizations should prove useful in analysing finite-size effects in systems subjected to non-periodic boundary conditions.

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By properly splitting up the summation domain $Z \times Z$, we see that the sums defined in equations (1.1a) and (1.1b) are related to each other by

$$R(\alpha, \beta; x) = -\tfrac{1}{4} + \tfrac{1}{4} W(\alpha, \beta; x) + \tfrac{1}{2} \sum_{m=1}^{\infty} (-1)^{m\alpha} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2]$$

$$-\tfrac{1}{2} \sum_{m=1}^{\infty} (-1)^{m\beta} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2] \tag{1.2}$$

where $(\alpha, \beta) \in \{(1, 1), (0, 0), (1, 0), (0, 1)\}$. Thus, when $\alpha = \beta$ it is apparent that

$$R(\alpha, \alpha; x) = -\tfrac{1}{4} + \tfrac{1}{4} W(\alpha, \alpha; x).$$

In the following we shall consider (in view of equation (1.2)) the related and somewhat more general one-dimensional (1D) sum

$$S(\alpha; x) \equiv \sum_{m \in Z} \mathrm{e}^{\mathrm{i}\pi\alpha m} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2] \tag{1.3}$$

where α is a real number and $x > 0$. Thus, when $\alpha \in \{0, 1\}$, it is easy to see that

$$\sum_{m=1}^{\infty} (-1)^{\alpha m} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2] = -\tfrac{1}{2} + \tfrac{1}{2} S(\alpha; x). \tag{1.4}$$

In sections 2 and 4 we shall discuss and develop criteria for the convergence of the sums $S(\alpha; x)$ and $W(\alpha, \beta; x)$, where α and β are real numbers. Then in sections 3 and 4, respectively, representations for these sums will be derived. In the following an asterisk (*) is used only in conjunction with a k-summation, so that $(a_p)^*$ means that a_k is not included in the sequence of parameters (a_p) for the current value of the summation index k; thus, for example,

$$\Gamma((a_p)^* - a_k) \equiv \prod_{\substack{j=1 \\ j \neq k}}^{p} \Gamma(a_j - a_k)$$

where Γ is the gamma function. If $p = 0$ in such a product, it is empty and reduces to unity.

The 1D sums in equation (1.2) are generalizations of Schlömilch series to which they reduce when $p = 0$. Although these sums of generalized hypergeometric functions have been evaluated in [8] for $p = 1$ and in [10] for $p \geqslant 1$, we shall derive (as already mentioned) a representation for $S(\alpha; x)$ which gives via equation (1.4) the unified result for $\alpha \in \{0, 1\}$, $x > 0$:

$$\sum_{m=1}^{\infty} (-1)^{m\alpha} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2] = -\frac{1}{2} + \frac{\sqrt{\pi}}{2x} \frac{\Gamma((b_{p+1}))}{\Gamma((a_p))} \frac{\Gamma((a_p) - \frac{1}{2})}{\Gamma((b_{p+1}) - \frac{1}{2})}$$

$$\times \sum_{\substack{m \in Z \\ (\alpha+2m)^2\pi^2 \leqslant 4x^2}} {}_{p+1}F_p \left[\begin{matrix} \frac{3}{2} - (b_{p+1}) \\ \frac{3}{2} - (a_p) \end{matrix} ; \frac{(\alpha + 2m)^2\pi^2}{4x^2} \right]$$

$$+ \frac{\sqrt{\pi}\,\Gamma((b_{p+1}))}{\Gamma((a_p))} \sum_{k=1}^{p} \frac{\Gamma(\frac{1}{2} - a_k)\Gamma((a_p)^* - a_k)}{\Gamma((b_{p+1}) - a_k)} \left(\frac{1}{4x^2} \right)^{a_k}$$

$$\times \sum_{\substack{m \in Z \\ (\alpha+2m)^2\pi^2 \leqslant 4x^2}} ((\alpha + 2m)^2\pi^2)^{a_k - \frac{1}{2}}$$

$$\times {}_{p+1}F_p \left[\begin{matrix} 1 + a_k - (b_{p+1}) \\ \frac{1}{2} + a_k, 1 + a_k - (a_p)^* \end{matrix} ; \frac{(\alpha + 2m)^2\pi^2}{4x^2} \right] \tag{1.5}$$

where convergence of the sum and generalized hypergeometric functions are determined by lemma 1 (see section 2). Thus, when $\alpha = 1$, equation (1.5) is valid for $(\ell - \frac{1}{2})\pi < x < (\ell + \frac{1}{2})\pi$, where ℓ is a non-negative integer, provided that for $1 \leqslant k \leqslant p$

$$\mathrm{Re}(a_k) > 0 \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2}$$

and if $x = (\ell + \frac{1}{2})\pi$, then

$$\mathrm{Re}(a_k) > 0 \qquad \mathrm{Re}(\Delta) > \tfrac{3}{2}.$$

When $\alpha = 0$, equation (1.5) is valid for $\ell\pi < x < (\ell + 1)\pi$, provided that

$$\mathrm{Re}(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2}$$

and if $x = (\ell + 1)\pi$, then

$$\mathrm{Re}(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}(\Delta) > \tfrac{3}{2}$$

where Δ is given by equation (1.1*c*).

The result given by equation (1.5) is slightly more general than the theorem in [10] (equations (3.3) and (3.4) therein), since it includes the cases $x = (\ell + 1)\pi$ and $(\ell + \frac{1}{2})\pi$ corresponding, respectively, to $\alpha = 0$ and 1. Furthermore, we tacitly assume values of the parameters for which expressions make sense. And we adapt throughout the convention that when the upper limit of a summation is less than the initial value of the lower limit, then the summation vanishes.

2. Convergence of the sums $S(\alpha; x)$ and $W(\alpha, \beta; x)$

We shall need the asymptotic result (see [9, equation (2.2*a*)]) for generalized hypergeometric functions:

$$ {}_pF_{p+1}[(a_p); (b_{p+1}); -z^2] = \left\{ \sum_{k=1}^{p} A_k \left(\frac{1}{z^2}\right)^{a_k} + A_{p+1} \left(\frac{1}{z^2}\right)^{\eta} \cos\left[2z - \pi\eta + \mathcal{O}\left(\frac{1}{z}\right)\right] \right\} \left[1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right] \qquad (2.1a)$$

where $|z| \to \infty$, $|\arg z| < \frac{1}{2}\pi$, the A_k $(1 \leqslant k \leqslant p + 1)$ are constants dependent on the parameters of the generalized hypergeometric function, and

$$\eta \equiv \tfrac{1}{2}\left(\Delta - \tfrac{1}{2}\right) \qquad (2.1b)$$

where Δ is given by equation (1.1*c*).

Now, without loss of generality, since α is an arbitrary real number, we may examine the convergence of either end of the bilateral series in equation (1.3) that defines $S(\alpha; x)$. Thus, by setting $z = mx$ in equation (2.1*a*), multiplying both sides of the latter by $\exp(\mathrm{i}\pi\alpha m)$, and for some sufficiently large fixed integer M, summing the result over $m \geqslant M$, we have for $x > 0$

$$\sum_{m \geqslant M} \mathrm{e}^{\mathrm{i}\pi\alpha m} {}_pF_{p+1}[(a_p); (b_{p+1}); -m^2x^2] = \left\{ \sum_{k=1}^{p} \frac{A_k}{x^{2a_k}} \sum_{m \geqslant M} \frac{\mathrm{e}^{\mathrm{i}\pi\alpha m}}{m^{2a_k}} + \frac{1}{2}\frac{A_{p+1}}{x^{2\eta}} \left(\mathrm{e}^{-\mathrm{i}\omega} \sum_{m \geqslant M} \frac{\mathrm{e}^{\mathrm{i}(\pi\alpha + 2x)m}}{m^{2\eta}} + \mathrm{e}^{\mathrm{i}\omega} \sum_{m \geqslant M} \frac{\mathrm{e}^{\mathrm{i}(\pi\alpha - 2x)m}}{m^{2\eta}} \right) \right\} \left[1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right] \qquad (2.2)$$

where $\omega \equiv \pi\eta - \mathcal{O}(1/x)$ and η is given by equation (2.1*b*).

We observe that the convergence of $S(\alpha; x)$ is determined by the convergence of the three m-summations in equation (2.2), all of which have the form $\sum_m \exp(\mathrm{i}\pi\beta m)m^{-z}$, where β is a real number. This sum converges for $\mathrm{Re}(z) > 1$, when β is a multiple of 2; otherwise it converges for $\mathrm{Re}(z) > 0$ (see [4, pp 59–60]). Thus, we deduce

Lemma 1. *For real α and $x > 0$, the sum $S(\alpha; x)$ converges under the conditions of each of the following four cases where $1 \leqslant k \leqslant p$.*

(a) If α is not a multiple of 2, and $\pi\alpha \pm 2x$ are not multiples of 2π, then

$$\mathrm{Re}(a_k) > 0 \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2}.$$

(b) If α is a multiple of 2, and $\pi\alpha \pm 2x$ are not multiples of 2π, then

$$\mathrm{Re}(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2}.$$

(c) If α is not a multiple of 2, and one of $\pi\alpha \pm 2x$ is a multiple of 2π, then

$$\mathrm{Re}(a_k) > 0 \qquad \mathrm{Re}(\Delta) > \tfrac{3}{2}.$$

(d) If α is a multiple of 2, and one of $\pi\alpha \pm 2x$ is a multiple of 2π, then

$$\mathrm{Re}(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}(\Delta) > \tfrac{3}{2}.$$

Furthermore, for real α and $x > 0$, the sum $S(\alpha; x)$ converges absolutely provided that

$$\mathrm{Re}(a_k) > \tfrac{1}{2} \qquad \mathrm{Re}(\Delta) > \tfrac{3}{2}$$

where Δ is given by equation (1.1c).

In a similar manner, we consider the convergence of the doubly infinite double sum $W(\alpha, \beta; x)$ defined by equation (1.1a). If now α and β are real numbers, by setting $z = x\sqrt{m^2 + n^2}$ in equation (2.1a), multiplying both sides of the latter by $\exp(\mathrm{i}\pi\alpha m + \mathrm{i}\pi\beta n)$, and for some sufficiently large fixed integer M, summing the result over $m \geqslant M$ and $n \geqslant M$, we have for $x > 0$

$$\sum_{m,n\geqslant M} \mathrm{e}^{\mathrm{i}\pi(\alpha m+\beta n)}\,{}_pF_{p+1}[(a_p); (b_{p+1}); -(m^2+n^2)x^2]$$
$$= \left\{ \sum_{k=1}^{p} \frac{A_k}{x^{2a_k}} \sum_{m,n\geqslant M} \frac{\mathrm{e}^{\mathrm{i}\pi(\alpha m+\beta n)}}{(m^2+n^2)^{a_k}} + \frac{1}{2}\frac{A_{p+1}}{x^{2\eta}} \left(\mathrm{e}^{-\mathrm{i}\omega} \sum_{m,n\geqslant M} \frac{\mathrm{e}^{\mathrm{i}(\pi\alpha m+\pi\beta n+2x\sqrt{m^2+n^2})}}{(m^2+n^2)^{\eta}} \right.\right.$$
$$\left.\left. + \mathrm{e}^{\mathrm{i}\omega} \sum_{m,n\geqslant M} \frac{\mathrm{e}^{\mathrm{i}(\pi\alpha m+\pi\beta n-2x\sqrt{m^2+n^2})}}{(m^2+n^2)^{\eta}} \right) \right\} \left[1 + \mathcal{O}\left(\frac{1}{x^2}\right) \right] \qquad (2.3a)$$

where $\omega = \pi\eta - \mathcal{O}(1/x)$ and η is given by equation (2.1b). Since α and β are arbitrary real numbers, we see without loss of generality that the convergence of $W(\alpha, \beta; x)$ is determined by the convergence of the three m, n-summations in equation (2.3a).

The third sum need not be considered separately so for conciseness, we denote the first and second m, n-summations on the right-hand side of equation (2.3a) by S and T, respectively, where in the sum T values of $x \neq 0$ are real. Obviously, necessary conditions that S and T converge, respectively, are

$$\mathrm{Re}(a_k) > 0 \quad (1 \leqslant k \leqslant p) \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2} \qquad (2.3b)$$

the latter of which follows from equation (2.1b), since $\mathrm{Re}(\eta) > 0$. In fact, the inequalities $\mathrm{Re}(a_k) > 0$ $(1 \leqslant k \leqslant p)$ guarantee the ordinary convergence of S provided that both α and

β are not multiples of 2. This is easily verified by applying a theorem due to Hardy to the example given in [4, p 97] with $v_{m,n} \equiv (m^2+n^2)^{-\eta}$.

Furthermore, both S and T converge absolutely, respectively, provided that

$$\mathrm{Re}(a_k) > 1 \quad (1 \leqslant k \leqslant p) \qquad \mathrm{Re}(\Delta) > \tfrac{5}{2} \tag{2.3c}$$

(see p 86, example 3 of [4] and p 52 of [12]), the latter of which follows from equation (2.1b), since $\mathrm{Re}(\eta) > 1$.

However, determination of the ordinary convergence of T (and thus also $W(\alpha, \beta; x)$) is problematic. Indeed, Borwein and Borwein have concluded in [3] that even demonstrating ordinary convergence for certain specializations of T 'involves unresolved questions of a deep and delicate number-theoretic nature'. However, since we shall be able to glean additional information (albeit heuristic in nature) from the development and actual representation for $W(\alpha, \beta; x)$ deduced in section 4, we shall conclude the analysis of the convergence criteria for $W(\alpha, \beta; x)$ in that section.

3. Representation for $S(\alpha; x)$

In the following we shall utilize the integral representation for ${}_pF_{p+1}[-\frac{1}{4}t^2]$ given by

$$\begin{aligned}
&{}_pF_{p+1}\left[(a_p);(b_{p+1});-\tfrac{1}{4}t^2\right] \\
&\quad = \frac{2}{\sqrt{\pi}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{1}{2})}{\Gamma((b_{p+1})-\frac{1}{2})}\int_0^1 \cos(tx)\,{}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(b_{p+1})\,; \\ \frac{3}{2}-(a_p)\quad ;\end{matrix}\; x^2\right]\mathrm{d}x \\
&\qquad + \frac{2}{\sqrt{\pi}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)} \\
&\qquad \times \int_0^1 (x^2)^{a_k-1/2}\cos(tx)\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}) & ; \\ \frac{1}{2}+a_k, 1+a_k-(a_p)^* & ;\end{matrix}\; x^2\right]\mathrm{d}x
\end{aligned} \tag{3.1a}$$

where t is a real number and for $1 \leqslant k \leqslant p$

$$\mathrm{Re}(a_k) > 0 \qquad \mathrm{Re}(\Delta) > \tfrac{1}{2}. \tag{3.1b}$$

This result is easily obtained by computing the inverse of the cosine transform of ${}_pF_{p+1}[-x^2]$ given in [10, lemma 1].

By making simple transformations, equation (3.1a) may be written as

$$\begin{aligned}
{}_pF_{p+1}[(a_p);(b_{p+1});-t^2x^2] &= \frac{1}{2\sqrt{\pi}}\frac{1}{x}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{1}{2})}{\Gamma((b_{p+1})-\frac{1}{2})} \\
&\quad \times \int_{\xi^2 \leqslant 4x^2} \mathrm{e}^{-\mathrm{i}t\xi}\,{}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(b_{p+1})\,; \\ \frac{3}{2}-(a_p)\quad ;\end{matrix}\; \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi \\
&\quad + \frac{1}{\sqrt{\pi}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k} \\
&\quad \times \int_{\xi^2 \leqslant 4x^2} \mathrm{e}^{-\mathrm{i}t\xi}(\xi^2)^{a_k-\frac{1}{2}}\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}) & ; \\ \frac{1}{2}+a_k, 1+a_k-(a_p)^* & ;\end{matrix}\; \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi
\end{aligned} \tag{3.2}$$

where $x > 0$, the integrations are over the interval $-2x \leqslant \xi \leqslant 2x$, and the conditional inequalities (3.1b) hold true.

Next, in equation (3.2) by setting $t = m$ and inserting the result for ${}_pF_{p+1}[-m^2x^2]$ into equation (1.3) we obtain

$$S(\alpha; x) = \frac{1}{2\sqrt{\pi}}\frac{1}{x}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{1}{2})}{\Gamma((b_{p+1})-\frac{1}{2})}$$
$$\times\int_{\xi^2\leqslant 4x^2}\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(\pi\alpha-\xi)m}\,{}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(b_{p+1})\;;\\ \frac{3}{2}-(a_p)\quad;\end{matrix}\;\frac{\xi^2}{4x^2}\right]\mathrm{d}\xi$$
$$+\frac{1}{\sqrt{\pi}}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k}$$
$$\times\int_{\xi^2\leqslant 4x^2}\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(\pi\alpha-\xi)m}(\xi^2)^{a_k-\frac{1}{2}}\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\;;\\ \frac{1}{2}+a_k,\,1+a_k-(a_p)^*\;;\end{matrix}\;\frac{\xi^2}{4x^2}\right]\mathrm{d}\xi \tag{3.3}$$

where the order of summation and integration have been interchanged in each term. The m-summations in this result may be rewritten by employing the identity for real μ:

$$\sum_{m\in Z}\mathrm{e}^{\mathrm{i}\mu m} = 2\pi\sum_{m\in Z}\delta(\mu-2\pi m) \tag{3.4}$$

(see [13], p 189, equation (17)), where δ is the delta function (or functional). Now setting $\mu = \pi\alpha - \xi$, replacing each sum in equation (3.3) by the right-hand side of equation (3.4), interchanging again the order of summation and integration in both terms, we have immediately upon performing the required formal term-by-term integrations

Theorem 1. *For $x > 0$ and real numbers α*

$$\sum_{m\in Z}\mathrm{e}^{\mathrm{i}\pi\alpha m}\,{}_pF_{p+1}[(a_p);(b_{p+1});-m^2x^2] = \frac{\sqrt{\pi}}{x}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\frac{\Gamma((a_p)-\frac{1}{2})}{\Gamma((b_{p+1})-\frac{1}{2})}$$
$$\times\sum_{m\in Z}^{(\alpha+2m)^2\pi^2\leqslant 4x^2}{}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(b_{p+1})\;;\\ \frac{3}{2}-(a_p)\quad;\end{matrix}\;\frac{(\alpha+2m)^2\pi^2}{4x^2}\right]$$
$$+2\sqrt{\pi}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k}$$
$$\times\sum_{m\in Z}^{(\alpha+2m)^2\pi^2\leqslant 4x^2}((\alpha+2m)^2\pi^2)^{a_k-1/2}$$
$$\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\;;\\ \frac{1}{2}+a_k,\,1+a_k-(a_p)^*\;;\end{matrix}\;\frac{(\alpha+2m)^2\pi^2}{4x^2}\right]. \tag{3.5}$$

Convergence of the doubly infinite sum on the left-hand side of equation (3.5) and the finite number of generalized hypergeometric functions on the right-hand side are guaranteed by the convergence criteria given in lemma 1. We mention that by applying equation (3.4) to equation (3.3), we have essentially made use of a form of the 1D Poisson summation formula (see [13, p 189]).

4. Representation for $W(\alpha, \beta; x)$

Similarly, we shall employ the 2D Poisson summation formula to obtain a closed-form representation for $W(\alpha, \beta; x)$. To this end, we shall have to evaluate the 2D Fourier transform $\mathcal{F}$ of the generalized hypergeometric function ${}_pF_{p+1}[-t^2(x^2+y^2)]$, where $t > 0$:

$$\begin{aligned}
&\mathcal{F}\{{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2)]\}\\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{\mathrm{i}\omega y}\,{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2)]\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \int_0^{\infty} \sigma\,{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2\sigma^2]\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\sigma(\xi\cos\theta+\omega\sin\theta)}\,\mathrm{d}\theta\,\mathrm{d}\sigma
\end{aligned}$$

which results from the polar coordinate transformation $x = \sigma\cos\theta$, $y = \sigma\sin\theta$. Furthermore, since

$$J_0(\sigma\sqrt{\xi^2+\omega^2}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\sigma(\xi\cos\theta+\omega\sin\theta)}\,\mathrm{d}\theta$$

we have

$$\begin{aligned}
&\mathcal{F}\{{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2)]\}\\
&\quad= 2\pi\int_0^{\infty} \sigma J_0(\sigma\sqrt{\xi^2+\omega^2})\,{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2\sigma^2]\,\mathrm{d}\sigma. \qquad (4.1a)
\end{aligned}$$

The discontinuous integral in equation (4.1a) exists provided that for $1 \leqslant k \leqslant p$

$$\mathrm{Re}(a_k) > \tfrac{1}{4} \qquad \mathrm{Re}(\Delta) > 1 \qquad (4.1b)$$

and may be evaluated by using [9, equations (4.4)]. Thus we see that the 2D Fourier transform of ${}_pF_{p+1}[-t^2(x^2+y^2)]$ vanishes when $\xi^2+\omega^2 > 4t^2$; otherwise when $\xi^2+\omega^2 < 4t^2$

$$\begin{aligned}
\mathcal{F}\{{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2)]\} &= \frac{\pi}{t^2}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)}\,{}_{p+1}F_p\left[\begin{matrix}2-(b_{p+1})\;;\\ 2-(a_p)\quad;\end{matrix}\;\frac{\xi^2+\omega^2}{4t^2}\right]\\
&+\frac{4\pi}{\xi^2+\omega^2}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\\
&\times\left(\frac{\xi^2+\omega^2}{4t^2}\right)^{a_k}{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\quad;\\ a_k, 1+a_k-(a_p)^*\;;\end{matrix}\;\frac{\xi^2+\omega^2}{4t^2}\right]
\end{aligned}$$

where the inequalities (4.1b) hold true.

Inversion of the Fourier transform given above now yields for $t > 0$

$$\begin{aligned}
{}_pF_{p+1}[(a_p); (b_{p+1}); -t^2(x^2+y^2)] &= \frac{1}{4\pi}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)}\frac{1}{t^2}\\
&\times\iint_{\xi^2+\omega^2\leqslant 4t^2} \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{e}^{-\mathrm{i}y\omega}\,{}_{p+1}F_p\left[\begin{matrix}2-(b_{p+1})\;;\\ 2-(a_p)\quad;\end{matrix}\;\frac{\xi^2+\omega^2}{4t^2}\right]\mathrm{d}\xi\,\mathrm{d}\omega\\
&+\frac{1}{\pi}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4t^2}\right)^{a_k}\\
&\times\iint_{\xi^2+\omega^2\leqslant 4t^2} \mathrm{e}^{-\mathrm{i}x\xi}\mathrm{e}^{-\mathrm{i}y\omega}(\xi^2+\omega^2)^{a_k-1}\\
&\times{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\quad;\\ a_k, 1+a_k-(a_p)^*\;;\end{matrix}\;\frac{\xi^2+\omega^2}{4t^2}\right]\mathrm{d}\xi\,\mathrm{d}\omega \qquad (4.2a)
\end{aligned}$$

where for $1 \leqslant k \leqslant p$

$$\mathrm{Re}(a_k) > \tfrac{1}{4} \qquad \mathrm{Re}(\Delta) > 1 \tag{4.2b}$$

which guarantee the convergence of both double integrals.

In equation (4.2*a*) replace x by m, y by n, then set $t = x$, multiply both sides by $\exp \mathrm{i}\pi(\alpha m+\beta n)$ and sum the resulting equation over $m \in Z, n \in Z$. Thus, for real α by defining $(-1)^{m\alpha} \equiv \exp(\mathrm{i}\pi\alpha m)$ and recalling the definition of $W(\alpha, \beta; x)$ given by equation (1.1*a*), we have for $x > 0$

$$\begin{aligned} W(\alpha,\beta;x) &= \frac{1}{4\pi}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)}\frac{1}{x^2}\iint_{\xi^2+\omega^2\leqslant 4x^2} {}_{p+1}F_p\left[\begin{matrix}2-(b_{p+1})\ ; & \dfrac{\xi^2+\omega^2}{4x^2}\\ 2-(a_p)\quad ; & \end{matrix}\right] \\ &\quad\times\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(\pi\alpha-\xi)m}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}(\pi\beta-\omega)n}\,\mathrm{d}\xi\,\mathrm{d}\omega+\frac{1}{\pi}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))} \\ &\quad\times\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k}\iint_{\xi^2+\omega^2\leqslant 4x^2}(\xi^2+\omega^2)^{a_k-1} \\ &\quad\times {}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\quad ; & \dfrac{\xi^2+\omega^2}{4x^2}\\ a_k, 1+a_k-(a_p)^*\ ; & \end{matrix}\right]\sum_{m\in Z}\mathrm{e}^{\mathrm{i}(\pi\alpha-\xi)m}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}(\pi\beta-\omega)n}\,\mathrm{d}\xi\,\mathrm{d}\omega \end{aligned} \tag{4.3}$$

where the order of summations and integrations have been interchanged in both terms. Now by employing equation (3.4) with $\mu = \pi\alpha - \xi$ and $\mu = \pi\beta - \omega$, respectively, so that each summation of exponentials in equation (4.3) is replaced by a summation of delta functions, and again in both terms interchanging the order of integrations and summations we obtain

$$\begin{aligned} W(\alpha,\beta;x) &= \frac{\pi}{x^2}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)}\sum_{m\in Z}\sum_{n\in Z}\iint_{\xi^2+\omega^2\leqslant 4x^2} {}_{p+1}F_p\left[\begin{matrix}2-(b_{p+1})\ ; & \dfrac{\xi^2+\omega^2}{4x^2}\\ 2-(a_p)\quad ; & \end{matrix}\right] \\ &\quad\times\delta(\pi\alpha-\xi-2\pi m)\delta(\pi\beta-\omega-2\pi n)\,\mathrm{d}\xi\,\mathrm{d}\omega \\ &\quad+4\pi\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{1}{4x^2}\right)^{a_k} \\ &\quad\times\sum_{m\in Z}\sum_{n\in Z}\iint_{\xi^2+\omega^2\leqslant 4x^2}(\xi^2+\omega^2)^{a_k-1}{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1})\quad ; & \dfrac{\xi^2+\omega^2}{4x^2}\\ a_k, 1+a_k-(a_p)^*\ ; & \end{matrix}\right] \\ &\quad\times\delta(\pi\alpha-\xi-2\pi m)\delta(\pi\beta-\omega-2\pi n)\,\mathrm{d}\xi\,\mathrm{d}\omega. \end{aligned}$$

Finally, on performing the required formal term-by-term integrations with regard to properties of the delta function we deduce

Theorem 2. *For $x > 0$, real numbers α and β*

$$\begin{aligned} \sum_{m\in Z}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}\pi\alpha m}\mathrm{e}^{\mathrm{i}\pi\beta n}{}_pF_{p+1}[(a_p);(b_{p+1});-x^2(m^2+n^2)] &= \frac{\pi}{x^2}\frac{\prod_{k=1}^{p+1}(b_k-1)}{\prod_{k=1}^{p}(a_k-1)} \\ &\quad\times\sum_{m\in Z,n\in Z}^{(\alpha+2m)^2+(\beta+2n)^2\leqslant 4x^2/\pi^2}{}_{p+1}F_p\left[\begin{matrix}2-(b_{p+1})\ ; & \dfrac{\pi^2}{4x^2}((\alpha+2m)^2+(\beta+2n)^2)\\ 2-(a_p)\quad ; & \end{matrix}\right] \\ &\quad+\frac{4}{\pi}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\sum_{k=1}^{p}\frac{\Gamma(1-a_k)\Gamma((a_p)^*-a_k)}{\Gamma((b_{p+1})-a_k)}\left(\frac{\pi^2}{4x^2}\right)^{a_k} \end{aligned}$$

$$\times \sum_{m\in Z,n\in Z}^{(\alpha+2m)^2+(\beta+2n)^2\leqslant 4x^2/\pi^2} ((\alpha+2m)^2+(\beta+2n)^2)^{a_k-1}$$

$$\times {}_{p+1}F_p\left[\begin{matrix} 1+a_k-(b_{p+1}) & ; \\ a_k, 1+a_k-(a_p)^* & ; \end{matrix} \frac{\pi^2}{4x^2}((\alpha+2m)^2+(\beta+2n)^2)\right]. \qquad (4.4)$$

Since both sides of equation (4.4) are even functions of x, the result holds for all real $x \neq 0$. Convergence criteria for $W(\alpha, \beta; x)$ whose representation is given by the latter result are summarized in lemma 2 below.

The conditional inequalities (4.2*b*) are evidently necessary for the convergence of $W(\alpha, \beta; x)$, since they are not only consistent, but also somewhat stronger than the necessary conditions for convergence of $W(\alpha, \beta; x)$ given by the conditional inequalities (2.3*b*). We also note that if x is such that the equality holds for a pair of integers (m, n) in the upper limit of the m, n-summations on the right-hand side of equation (4.4), then in order to ensure convergence of the generalized hypergeometric series ${}_{p+1}F_p$ [1], we need only require $\text{Re}(\Delta) > 2$. This is consistent with the latter of the slightly stronger conditional inequalities (2.3*c*) which provide for the absolute convergence of the sum T in equation (2.3*a*). These observations together with those noted in section 2 are now summarized below by using the strongest applicable inequalities. Thus we have

Conjectural lemma 2. *For real α, β, and $x > 0$, the sum $W(\alpha, \beta; x)$ converges under the conditions of each of the following four cases where $1 \leqslant k \leqslant p$.*

(a) If α and β are not multiples of 2, and $(\alpha+2m)^2+(\beta+2n)^2 < 4x^2/\pi^2$, then

$$\text{Re}(a_k) > \tfrac{1}{4} \qquad \text{Re}(\Delta) > 1.$$

(b) If one of α and β is a multiple of 2, and $(\alpha+2m)^2+(\beta+2n)^2 < 4x^2/\pi^2$, then

$$\text{Re}(a_k) > 1 \qquad \text{Re}(\Delta) > 1.$$

(c) If α and β are not multiples of 2, and $(\alpha+2m)^2+(\beta+2n)^2 \leqslant 4x^2/\pi^2$, then

$$\text{Re}(a_k) > \tfrac{1}{4} \qquad \text{Re}(\Delta) > 2.$$

(d) If one of α and β is a multiple of 2, and $(\alpha+2m)^2+(\beta+2n)^2 \leqslant 4x^2/\pi^2$, then

$$\text{Re}(a_k) > 1 \qquad \text{Re}(\Delta) > 2.$$

Furthermore, for real α, β, and $x > 0$, the sum $W(\alpha, \beta; x)$ converges absolutely provided that

$$\text{Re}(a_k) > 1 \qquad \text{Re}(\Delta) > \tfrac{5}{2}$$

where Δ is given by equation (1.1c).

In conclusion, we emphasize that since lemma 2 is based in part on heuristic and formal methods, it is conjectural in nature. Moreover, a rigorous proof may involve (as mentioned earlier in section 2) heretofore unresolved questions which, it is hoped, may stimulate further research.

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